Observations on the Strained Coordinate Method for Transonic Flows

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Introduction

RECENTLY papers[1,2] concerning a strained coordinate method for discontinuous transonic flow problems have appeared in the literature. In particular, the application of this technique for a linear interpolation or extrapolation between two known results has raised some questions regarding the relationship of the strained coordinate technique to normal interpolation procedures. Some confusion also seems to have occurred regarding the treatment of the Oswatitsch-Zierep[3] "shock foot singularity." It is the purpose of this Note to examine these questions and (hopefully) finally resolve them.

Analysis

In Fig. 1 a typical velocity distribution for transonic flow $u(x)$ is sketched. This is the curve ABCD. A shock wave is present, denoted by the line BC, with the shock foot singularity at the point C. At C, the derivative $\partial u/\partial x$ is logarithmically singular. A second transonic flow, related by some small parameter change to the first, is represented by the curve ABCD. The purpose of a linear interpolation procedure is to obtain a third solution, A B″ C″D from the first two (known) solutions.

Let the difference between the ABCD and AB′C′D solution be characterized by the small parameter ϵ_0 and the difference between AB″C″D and ABCD characterized by ϵ. The shock locations C, C′, C″ are denoted by x_s', $\bar{x}_s$, x_s. In both the strained coordinate method and linear interpolation the new shock location is

$$x_s = x_s' + (\epsilon/\epsilon_0)(\bar{x}_s - x_s') \tag{1}$$

Consider now the velocity distribution $u(x)$ in the region AB where no discontinuities or singularities appear. The strained coordinate method gives[2]

$$u(x) = u_0(x') + (\epsilon/\epsilon_0)[u_1(\bar{x}) - u_0(x')] \tag{2}$$

where $u_0(x')$ refers to ABCD, $u_1(\bar{x})$ refers to AB′C′D. The physical coordinate is x' and

$$\bar{x} = x' + \delta x_s x_1(x') \tag{3}$$

$$x = x' + (\epsilon/\epsilon_0)\delta x_s x_1(x') \tag{4}$$

where

$$x_1(x') = \frac{x'(1-x')}{x_s'(1-x_s')} \tag{5a}$$

and

$$\delta x_s = \bar{x}_s - x_s' \tag{5b}$$

A normal linear interpolation procedure gives

$$u(x) = u_0(x) + (\epsilon/\epsilon_0)[u_1(x) - u_0(x)] \tag{6}$$

Now from Eqs. (3) and (4)

$$x = x' + (\epsilon/\epsilon_0)\delta x_s x_1(x')$$

$$x = \bar{x} + [(\epsilon/\epsilon_0) - 1]\delta x_s x_1(x')$$

and thus, by a Taylor series expansion,

$$u_0(x) = u_0(x') + \frac{\epsilon}{\epsilon_0}\delta x_s x_1(x')\frac{\partial u_0}{\partial x'} + \ldots$$

$$u_1(x) = u_1(\bar{x}) + \left(\frac{\epsilon}{\epsilon_0} - 1\right)\delta x_s x_1(x')\frac{\partial u_1}{\partial \bar{x}} + \ldots \tag{7}$$

Substitution of Eq. (7) into Eq. (6) and retaining only linear terms in δx_s and ϵ gives

$$u(x) = u_0(x') + \frac{\epsilon}{\epsilon_0}\left(\frac{\epsilon}{\epsilon_0} - 1\right)\delta x_s x_1(x)\left[\frac{\partial u}{\partial \bar{x}} - \frac{\partial u_0}{\partial x'}\right]$$

$$+ \frac{\epsilon}{\epsilon_0}[u_1(\bar{x}) - u_0(x')] \tag{8}$$

If, as will be the case in region AB,

$$\left[\frac{\partial u_1}{\partial \bar{x}} - \frac{\partial u_0}{\partial x'}\right] \sim O(\epsilon) \tag{9}$$

then the second term on the right of Eq. (8) can be neglected and it may be seen that the strained coordinate method and linear interpolation are equivalent. A similar equivalence occurs in the region C′D provided the point in question is not too close to the singularity at C′. This would make $\partial u_1/\partial \bar{x}$ large compared with $\partial u_0/\partial x'$ and would invalidate the assumption of Eq. (9).

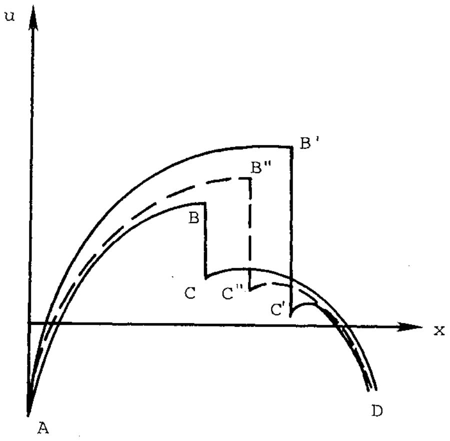

Fig. 1 Sketch of transonic velocity distribution on an airfoil.

Received July 13, 1979; revision received Sept. 5, 1979.
Index categories: Transonic Flow; Analytical and Numerical Methods.
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Consider now interpolation or extrapolation in the region BB′. An extrapolation for $u_0(x')$ can be made using a Taylor series expansion about some point x'_p close to B. Thus,

$$u_0(x) = u_0(x'_p) + (x - x'_p)\left(\frac{\partial u_0}{\partial x'}\right)_{x_p} + \ldots \tag{10}$$

A similar analysis to that previously given leads to the result

$$\begin{aligned} u(x) &= u_0(x'_p) + \frac{\epsilon}{\epsilon_0}\left[u_1(\bar{x}) - u_0(x'_p)\right] \\ &+ \frac{\epsilon}{\epsilon_0}\left\{\left(\frac{\epsilon}{\epsilon_0} - 1\right)\delta x_s x_1(x')\frac{\partial u_1}{\partial \bar{x}}\right. \\ &\left. - \Delta x_p\left(\frac{\epsilon}{\epsilon_0} - 1\right)\left(\frac{\partial u_0}{\partial x'}\right)_{x_p}\right\} \end{aligned} \tag{11}$$

If $\partial u_1/\partial \bar{x}$ and $[\partial u_0/\partial x']_{x_p}$ are sufficiently close and if

$$\Delta x_p \sim O[\delta x_s x_1(x')]$$

then the third term in Eq. (11) is negligible and again the linear interpolation/extrapolation is equivalent to the strained coordinate method. A similar analysis can be applied to the region C″C′, in which case, if x'_p is close to C′, $[\partial u_1/\partial \bar{x}]_{\bar{x}_p}$ is very large and interpolation/extrapolation is not equivalent to the strained coordinate method since $[\partial u_1/\partial \bar{x}]_{x_p}$ would not be sufficiently close to the much more regular $\partial u_0/\partial x'$ in C″C′. Indeed, it is difficult to see how values in C″C′ can be obtained by linear interpolation/extrapolation since this would involve an expansion through the singularity at C′. It is probably possible, however, to obtain a solution in CC′ if the shock jump relations are used at B″ to get values at C″ and a higher order interpolation used in C′C″.

Consider now the behavior of the strained coordinate method at the shock foot. The behavior of the velocity at C, C′ is given by the form

$$\begin{aligned} u_1(\bar{x}) &= u_1(\bar{x}_{s\_}) + \alpha_1(\bar{x} - \bar{x}_s)\ln(\bar{x} - \bar{x}_s) \\ u_0(x') &= u_0(x'_{s\_}) + \alpha_0(x' - x'_s)\ln(x' - x'_s) \end{aligned} \tag{12}$$

where $u_1(\bar{x}_{s\_})$, $u_0(x'_{s\_})$ are values at B′ and B, respectively, and α_1, α_0 are constants depending on the shock strength and the airfoil curvature. Using the strained coordinate method, Eq. (2), gives for values just downstream of C″

$$\begin{aligned} u(x) &= u_0(x'_{s\_}) + \alpha_0(x' - x'_s)\ln(x - x'_s) \\ &+ (\epsilon/\epsilon_0)\{u_1(\bar{x}_{s\_}) - u_0(x'_{s\_}) + \alpha_1(\bar{x} - \bar{x}_s)\ln(\bar{x} - \bar{x}_s) \\ &- \alpha_0(x' - x'_s)\ln(x' - x'_s)\} \end{aligned} \tag{13}$$

From Eqs. (3) and (5) it can be seen that

$$(\bar{x} - \bar{x}_s) = x' - x'_s + \delta x_s(x_1(x') - 1)$$

If $(x' - x'_s) = \delta$ then, using Eq. (5),

$$(\bar{x} - \bar{x}_s) = \delta[1 + a(x')] \tag{14}$$

where

$$a(x') = \frac{\delta x_s(1 - x' - x'_s)}{x_s(1 - x_s)} \tag{15}$$

Substitution of Eq. (14) into Eq. (13) gives

$$\begin{aligned} u(x) &= u_0(x_{s\_}) + \alpha_0\delta\ln\delta + (\epsilon/\epsilon_0)\{u_1(\bar{x}_{s\_}) - u_0(x'_{s\_}) \\ &+ \alpha_1\delta[1 + a(x')]\ln\delta[1 + a(x')] - \alpha_0\delta\ln\delta\} \end{aligned} \tag{16}$$

Since δx_s is assumed small

$$\ln\delta[1 + a(x')] \sim \ln\delta$$

Then

$$\begin{aligned} u(x) &= \left\{u_0(x'_s) + \frac{\epsilon}{\epsilon_0}[u_1(\bar{x}_{s\_}) - u_0(x_{s\_})]\right\} \\ &+ \left(\alpha_0 + \frac{\epsilon}{\epsilon_0}\{\alpha_1[1 + a(x')] - \alpha_0\}\right)\delta\ln\delta \end{aligned} \tag{17}$$

The form of the singular part of $u(x)$ at C″ can be found by the above analysis to be

$$\alpha_2\delta[1 + (\epsilon/\epsilon_0)a(x')]\ln\delta[1 + (\epsilon/\epsilon_0)a(x')]$$

where α_2 is some constant.

Since $a(x')$ is small this can be reduced to

$$\alpha_2[\delta\ln\delta + (\delta\ln\delta)(\epsilon/\epsilon_0)a(x')]$$

Thus,

$$\begin{aligned} \delta\ln\delta &\simeq \delta[1 + (\epsilon/\epsilon_0)a(x')] \\ &\ln\{\delta[1 + (\epsilon/\epsilon_0)a(x')]\}[1 - (\epsilon/\epsilon_0)a(x')] \end{aligned}$$

Substitution in Eq. (13) gives

$$\begin{aligned} u(x) &= \{u_0(x'_s) + (\epsilon/\epsilon_0)[u_1(x_{s\_}) - u_0(x'_{s\_})]\} \\ &+ \left[\alpha_0 + (\epsilon/\epsilon_0)(\alpha_1 - \alpha_0) + (\epsilon/\epsilon_0)\left(1 - \frac{\epsilon}{\epsilon_0}\right)a(x')(\alpha_1 - \alpha_0)\right] \\ &\times \delta[1 + (\epsilon/\epsilon_0)a(x')]\ln\{\delta[1 + (\epsilon/\epsilon_0)a(x')]\} \end{aligned} \tag{18}$$

Since we are dealing in small perturbations it is obvious that $(\alpha_1 - \alpha_0)$ is small and, hence, the term ϵ/ϵ_0 $[1 - (\epsilon/\epsilon_0)]a(x')$ $(\alpha_1 - \alpha_0)$ can be neglected in Eq. (18). It can then be seen that the strained coordinate system not only transfers the shock to its correct location, from Eq. (1), but also introduces the correct shock foot singularity with a strength $[\alpha_0 + (\epsilon/\epsilon_0)(\alpha_1 - \alpha_0)]$, obtained by linear interpolation of the strengths at C, C′. Thus in the strained coordinate method the treatment of the shock foot singularity is consistent with the treatment of the smoother parts of the solution.

Conclusions

The strained coordinate interpolation method used for transonic flows has been compared with normal interpolation/extrapolation procedures. It is found that both methods are essentially equivalent in smooth regions of the solution. However, normal linear extrapolation does not appear to be applicable in the region just behind a shock wave where the acceleration is infinite. The strained coordinate method does move the shock and its associated shock foot singularity to the correct location and scales the strength of the singularity according to linear interpolation.

Acknowledgment

This work was sponsored under AFOSR Contract F49620-79-C-0053.

References

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